Direct Transcription Solution of Optimal Control Problems with Control Delays

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Abstract. The numerical solution of optimal control problems is important in many areas. Often the models for these problems have delays. Direct transcription is a popular approach for the numerical solution of optimal control problems in industry. However, much less work has been done on the direct transcription solution of optimal control problems with delays. This talk will describe progress and challenges in developing a general purpose industrial grade direct transcription code that can handle problems with delays. Of special interest will be the more challenging case of control delays.

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INTRODUCTION

Optimal control problems with delays appear throughout science and industry in problems ranging from chemical reactors [1] to the cardiovascular-respiratory system [2]. One of the popular approaches to solving optimal control problems is direct transcription [3]. Direct transcription is popular since it does not require forming or solving the necessary conditions and it is easy for the user to attempt to solve problems with a wide variety of constraints. Direct transcription proceeds by discretizing the problem on a coarse mesh and solving the resulting nonlinear programming problem on a series of algorithmically constructed finer meshes. Direct transcription can often solve problems that are not amenable to other methods. To do this it must use state of the art sparse nonlinear programming (NLP) solvers.

In spite of its popularity there has been little work done on direct transcription codes for problems with delays. Recently we have begun work on the construction of a general purpose direct transcription optimal control solver that can accommodate state and control delays and state and control constraints. This software package known as SOCX can already solve a variety of problems with state delays [4, 5]. However, the consideration of control delays is proving more challenging. This paper will focus on discussing our progress on treating problems with control delays and describe some of the remaining challenges.

Section will briefly describe our particular implementation. Then in Section we will discuss control delays. We will begin with a deceptively easy problem which will illustrate several issues that are unique to control delays. In the full talk and the full paper we will show how to solve this particular problem successfully and we will turn to the consideration of more general problems. Finally, conclusions are in Section .

DIRECT TRANSCRIPTION IMPLEMENTATION

The particular implementation that we are developing is called SOCX (Sparse Optimal Control software eXtended) and is available from Applied Mathematical Analysis, but the discussion here is relevant to any other direct transcription codes trying to solve similar problems.

Once constructed, the NLP problem on a particular grid is solved by either a sparse sequential quadratic program (SQP) method or a sparse barrier method. The other two key pieces are how the software sets up the NLP problem for a given grid and how the grid is refined between NLP solves. Along with these two topics there is the important role that problem formulation plays. This later consideration is very important in practice since the form of the cost function is something that is under the control of the user.

Currently there are two discretizations available, the trapezoid (TR) method and the Hermite-Simpson (HS). They are second and fourth order as integrators. The dynamic equations are allowed to be index one differential algebraic

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equations (DAEs) [6]. At each grid level we have a time grid and NLP variables which are the values of the controls and states at the grid points. In the case of HS there are also midpoint variables. All of these variables are constrained by the discretization.

Delayed quantities are handled as followed. Suppose there is a delayed variable $z(t-\tau)$ and it's undelayed version $z(t)$. The delay τ may vary with time and may be positive or negative but it may not change sign. The grid is $G = [t_0, \ldots, t_f]$. The software introduces a new algebraic variable $v(t)$ and a corresponding constraint on the grid to ensure that $v(t)$ equals the delayed variable z. In SOCX we refer to undifferentiated variables, including control variables, as algebraic variables. Then for every grid point t_i if $t_i - \tau < t_0$, then $v(t_i)$ is given by the prehistory of z and is just $z(t_i - \tau)$. If $t_i - \tau \geq t_0$, let $[t_k,\ t_{k+1})$ be the grid subinterval that contains $t_i - \tau$. Then an NLP constraint is added which says that v_i is a linear combination of z_k, z_{k+1} so that v_i is equal to an interpolated estimates of $z(t_i - \tau)$. At first glance this may seem like a lot of additional constraints but it is no more than what would be dealt with on a slightly higher dimensional dynamic system. However, these additional constraints do impact on the sparsity structure.

In the case of state only delays this procedure seems to be working fine as examples in [4, 5] and an unreported test set show. The situation with control delays is more delicate and has some impact on mesh selection.

Note that the resulting system is always a DAE. Even if we start with the ordinary differential equation

$$x' = f(x(t), x(t-r), u(t), u(t-s))$$

with state variables x, the software sets up the DAE

$$x' = f(x(t), y(t), u(t), v(t)) \tag{1a}$$

$$0 = y(t) - x(t-r) \tag{1b}$$

$$0 = v(t) - u(t-r) \tag{1c}$$

with state variables x, y.

CONTROL DELAYS

It is convenient in this abstract to use a particular example for purposes of discussion. Additional examples will be in the actual presentation and the longer version of this abstract. The longer version is expected to be published after the conference. The idea is to have as simple an example as possible so that it will be clear how specific problem characteristics affect algorithmic results. The problem is to minimize

$$J(u) = \int_0^5 x^2(t) + \alpha u^2(t) + \beta u^2(t-1)\, dt \tag{2}$$

subject to

$$x' = u(t-1), \tag{3a}$$

$$x(0) = 1 \tag{3b}$$

$$u(t) = 1 \quad \text{for } -1 \leq t < 0 \tag{3c}$$

on the interval $0 \leq t \leq 5$. Here α, β are parameters that will be used to show different behavior. Cost (2) is the type of quadratic cost that arises in many control applications.

In default operation SOCX starts with TR for the first couple of grids and then switches to HS. This has been found to be more robust during previous work on systems without delays. In this particular study to help us clearly identify what features have what effects we focus on just using TR.

Our software can take problems formulated as phases, that is formulated differently on different subintervals. For the $\alpha = 1, \beta = 0$ case,

$$J_1(u) = \int_0^5 x^2(t) + u^2(t)\, dt, \tag{4}$$

and using either phases or a method of steps (MOS) formulation [7, 8] we find that the solution is given in Figure 1. Note that since u has no effect on the state after $t = 4$, but does appear in the cost, that $u(t) = 0$ for $t > 4$.

The first time we ran the software which was working well for state delays on (3) with (4) we got Figure 2 which is

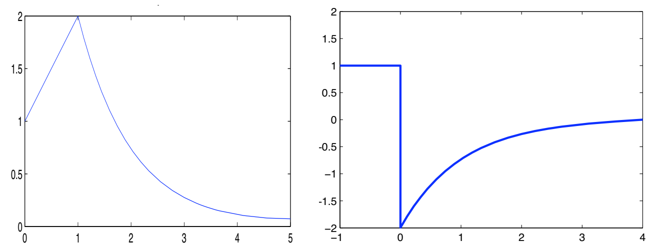


FIGURE 1. Optimal state x (left) and optimal control u (right) for (3) with cost (4).

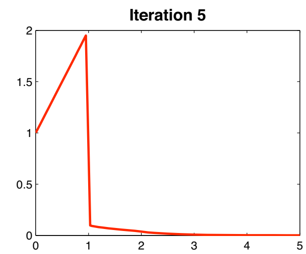


FIGURE 2. State solution on grid refinement 5 for (3) with cost (4).

clearly not correct. In Figure 2 and the other figures that follow we have shifted the v graph one unit to the left so we can more clearly compare it to the graph of u.

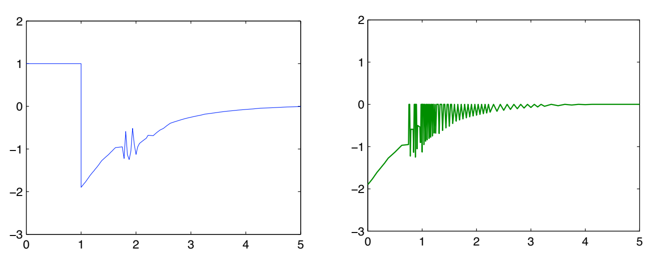
Note that $v(t) = u(t-1)$ is the actual control and not $u(t)$. Cost is often something that is at least partly at the discretion of the user. It is known that making sure the cost has the correct terms in it is important for nondelayed systems [9]. Accordingly we try the cost

$$\int_0^5 x(t)^2 + \frac{1}{2}(u(t)^2 + v(t)^2)dt \tag{5}$$

which includes v instead of (4). The results are in Figure 3 and 4.

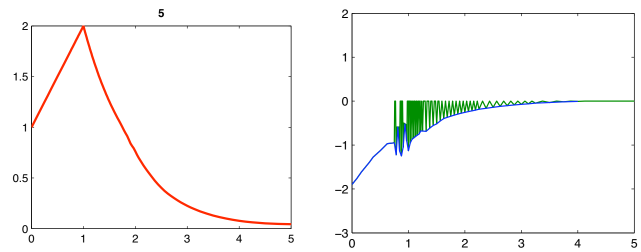
First we note that modifying the cost has had a dramatic effect. The state in Figure 4 now looks almost identical to that in Figure 1. We also see that the $v(t)$ graph in Figure 3 is very similar to the v graph in Figure 1 except for some chatter around 2 which is one delay period after 1. However, the u graph on the right side of Figure 3 oscillates widely. Looking at the right side of Figure 4 where u and v are plotted together we note that u is oscillating between being zero and being equal to $v(t+1)$.

This example is suggesting several things. For one, the correct cost formulation is crucial. The other is that the way that the interpolation is incorporated is also very important. These will be carefully examined in the full talk and paper



$v(t)$ $u(t)$

FIGURE 3. $v(t) = u(t-1)$ (shifted) and $u(t)$ for (3) with cost (5) on grid refinement 5.



$x(t)$ $\quad$ $u(t)$ green $v(t)$ blue

FIGURE 4. $x(t)$, $v(t) = u(t-1)$ (shifted), and $u(t)$ for (3) with cost (5) on grid refinement 5.

where a good direct transcription solution of this particular problem will be obtained.

CONCLUSION

The numerical solution of constrained optimal control problems with control and state delays is important in a number of industrial and scientific applications. Direct transcription is an important technique in optimal control but has not traditionally been used on systems with delays. This talk has discussed progress on designing a direct transcription optimal control package for solving problems with state and control constraints and delays.

A number of issues that arise with control delays have been illustrated with examples. Some successfully solved problems are given to illustrate the progress to date. Remaining open questions are discussed.

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